13 Nested Quantifiers (continued)

<u>Example</u> True / False: Justify your answers with an example or a short sentence.

- $\forall x \in \mathbb{R} : [\forall y \in \mathbb{R} : x + y = 1] \equiv$ "For all real #s x, for all real #'s y, $x+y=1$"
  $\equiv$ False: if $x=0, y=0$, then $x+y=0 \neq 1$.

- $\forall x \in \mathbb{R} : [\exists y \in \mathbb{R} : x + y = 1] \equiv$ "For all real #'s x, there exists a real # y such that $x+y=1$"
  $\equiv$ True: no matter the x, pick $y = 1 - x$, so
  $x + y = x + (1-x) = 1$.

- $\exists x \in \mathbb{R} : [\forall y \in \mathbb{R} : x + y = 1] \equiv$ "There exists a real # x such that for all real #'s y, $x+y=1$"
  $=$ False: if $\underline{x} + \underline{y} = 1$, then a different choice of y-value, namely $(1+\underline{y})$ gives
  $\underline{x} + (1 + \underline{y}) = \underline{x} + \underline{y} + 1 = 1 + 1 = 2 \neq 1$.

- $\exists x \in \mathbb{R} : [\exists y \in \mathbb{R} : x + y = 1] \equiv$ True: just let $x = 1, y = 0$.

<u>Example</u> Negate "There is a specialist who can solve every problem."

$\equiv \neg (\exists x \in \underbrace{S}_{\{\text{Specialists}\}} : [\forall y \in \underbrace{P}_{\{\text{problems}\}} : \text{"}x \text{ solves } y\text{"}])$

$\equiv \forall x \in S : [\exists y \in P : \text{"}x \text{ cannot solve } y\text{"}]$

$\equiv$ "For every specialist x, there is a problem y such that x cannot solve y."

$\equiv$ "Every specialist has a problem that they cannot solve."

<u>Interlude</u> How sets & logic relate: Logic specify the subset from the universe set:

$$\sim(\underbrace{S}_{\text{Students}} \cup \underbrace{F}_{\text{Faculty}}) = \{x \in \overbrace{R}^{\{\text{People at Rowan}\}} : \neg((x \in S) \vee (x \in F))\} \overset{\text{de Morgan}}{=} \{x \in R : (x \notin S) \wedge (x \notin F)\}$$

$$= (\sim S) \cap (\sim F)$$

If queried, these have different runtimes.

14 Functions

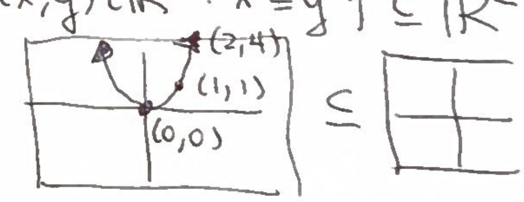
Sometimes we want to relate two sets. One way to do this is by drawing arrows from a first set (the domain) to a second set (the codomain): A function takes each element of the domain to a unique element of the codomain.

Ex 1 $f: \{0,1,2,3,\dots,255\} \to \{0,100,200\}$ is a "quantization" function which reduces the # of colors needed to store an RGB image.

$0 \to 0$
$1 \to 0$
$2 \to 0$
$\vdots$
$99 \to 0$
$100 \to 100$
$\vdots$
$199 \to 100$
$200 \to 200$
$\vdots$
$255 \to 200$

Ex 2 $\text{NOT}: \{T,F\} \to \{T,F\}$
$T \to F$
$F \to T$.

Ex 3 $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is associated to its graph $\{(x,y) \in \mathbb{R}^2 : x^2 = y\} \subseteq \mathbb{R}^2$
$1 \to 1^2 = 1$
$2 \to 2^2 = 4$
$-1 \to (-1)^2 = 1$



Ex 4 $g: \mathbb{R}^2 \to \mathbb{R}$, $g(x,y) = x+y$ is the "addition function".
$(1,3) \mapsto 1+3 = 4$
$(2,-5) \to 2+(-5) = -3$

Def'n Given function $f: A \to B$, $f(a)$ is the output of a, and the range of f is the set of all outputs of f:

$$\text{range}(f) = \{b \in B : b = f(a) \text{ for some } a \in A\}$$
$$= \{b \in B : [\exists a \in A : b = f(a)]\}.$$

So in Ex 3, 1 is the output of $x=+1$ and of $x=-1$, but -1 is not an output.

Def'n $f: A \to B$ is <u>onto</u> if $\text{range}(f) = B$, i.e.

for every $b \in B$, there exists $a \in A$ such that $b = f(a)$.

Ex 5 $f: \{0,1,2\} \to \{3,4\}$ is onto: every element of codomain $\{3,4\}$ is an output (pointed at).

$0 \to 3$
$1 \to 4$
$2 \to 4$

$g: \{0,1,2\} \to \{3,4\}$ is not onto: 4 isn't an output.

$0 \to 3$
$1 \to 3$
$2 \to 3$
4

> **Moral**: To show f isn't ~~an output,~~ onto, find something in codomain that's not an output.

Ex 6 $f: \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not onto: -1 isn't output bc $-1 = f(x) = x^2$ has no real # solution for x.

Ex 7 Show $f: \mathbb{Z} \to \mathbb{Z}$, $f(n) = 3n$ is not onto.

<u>A</u> Notice $f(-1) = -3$, $f(0) = 0$, $f(1) = 3$ seems to skip 1. So Show 1 isn't output: $f(x) = 3x = 1 \xrightarrow[\div 3]{\text{Solve for } x} x = \frac{1}{3} \notin \text{domain}(f) = \mathbb{Z}$ so f isn't onto.

> **Moral**: To show f is onto, show every y in codomain comes from / can be solved from an x in domain.

Ex 8 Show $g: \mathbb{R} \to \mathbb{R}$, $g(x) = 3 \cdot x$ is onto.

<u>A</u> Let $y \in \text{codomain}(g) = \mathbb{R}$. Then $y = 3x \xrightarrow[\div 3]{\text{Solve for } x} x = \frac{y}{3} \in \mathbb{R} = \text{domain}(g)$ so g is onto.

real # so $\div 3$ still makes it a real #.

Ex 9 Show $f: \mathbb{Z} \to \mathbb{Z}$, $f(x) = 2 - x$ is onto.

<u>A</u> Let $y \in \text{codomain}(f) = \mathbb{Z}$. Then $y = 2 - x \xrightarrow{\text{Solve for } x} x = 2 - y$ is still in $\mathbb{Z} = \text{domain}(f)$ so f is onto.

Note Ex 8 fails if we replace $\mathbb{R}$ by $\mathbb{Z}$ [both domain & codomain] bc integer like $y = 1$ divided by 3 need not be integer.